4) Wave Propagation/Flow Simulation.

  Consider a regular set of N square cells with dimension D_c and center points $\mathbf{r}_i=(x_i,y_i)$. At time t each cell holds water of thickness $H_i(t)=H(\mathbf{r}_i,t)$, with mean horizontal velocity $\mathbf{v}_i(t)=\mathbf{v}(\mathbf{r}_i,t)$, and mean horizontal acceleration $\mathbf{a}_i(t)=\mathbf{a}(\mathbf{r}_i,t)$ (Figure 2, top left). For dry locations $H_i(t)$ would be zero of course. The entire concept of wave propagation involves updating those conditions to time t+dt. There are many ways to this but we employ a variant of "tsunami balls" (*Ward* and *Day*, 20xx, 20yy).

  Pick one cell, say i=10 (red square, Figure 2 top right). With its known velocity and acceleration displace the water cell to a new center point

$$\tilde{\mathbf{r}}_i = \mathbf{r}_i(t) + \mathbf{v}_i(t)dt + 0.5\mathbf{a}_i(t)dt^2 \quad (1)$$

and give it a new mean velocity

$$\mathbf{v}_i = \mathbf{v}_i(t) + \mathbf{a}_j(t)dt \quad (2)$$

We wish to partition the volume and linear momentum of the water in the displaced cell among the N original cells. A bit of thinking tells you that the partitioned volume of the displaced i-th cell into the j-th original cell is

$$\delta V_{ji} = (H_iD_c^2)(1 - \frac{|\tilde{x}_i - x_j|}{D_c})(1 - \frac{|\tilde{y}_i - y_j|}{D_c})$$

$$if \quad \frac{|\tilde{x}_i - x_j|}{D_c} < 1 \ and \ \frac{|\tilde{y}_i - y_j|}{D_c} < 1; \quad (3)$$

$$otherwise \quad \delta V_{ji} = 0$$

Clearly there is no need to run the partitioning through all j=N cells because at most, only four original cells overlap the displaced one (Figure 2, bottom left). Moreover since $\tilde{\mathbf{r}}_i$ is known and the cells are square, it is simple to determine which four cells overlap.

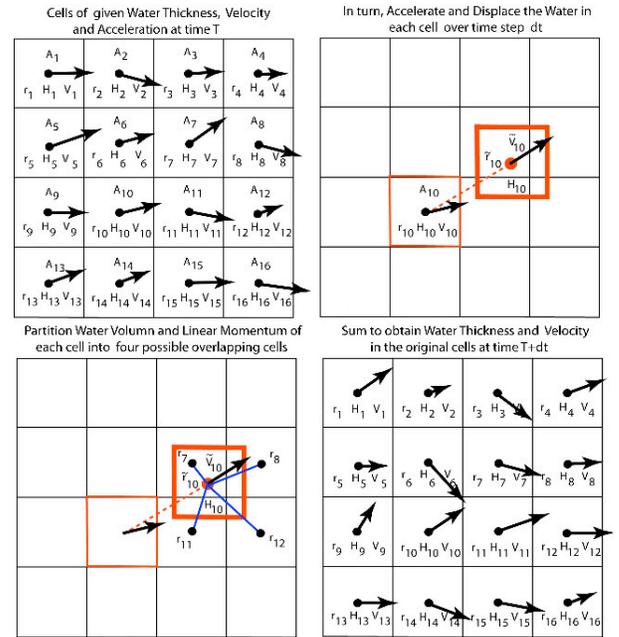 Partitioning of vector linear momentum follows in the same way

Figure 2. Wave propagation concept.

$$\delta\mathbf{M}_{ji} = (\rho_wH_iD_c^2\tilde{\mathbf{v}}_i)(1 - \frac{|\tilde{x}_i - x_j|}{D_c})(1 - \frac{|\tilde{y}_i - y_j|}{D_c})$$

$$if \quad \frac{|\tilde{x}_i - x_j|}{D_c} < 1 \ and \ \frac{|\tilde{y}_i - y_j|}{D_c} < 1; \quad (4)$$

$$otherwise \quad \delta\mathbf{M}_{ji} = 0$$

The N updated thickness and velocity values on the original grid come from summing and normalizing (3) and (4) over all i displaced cells

$$H_j(t + dt) = \frac{\sum_{i=1}^{N}\delta V_{ji}}{D_c^2} \quad (5)$$

$$\mathbf{v}_j(t + dt) = \frac{\sum_{i=1}^{N}\delta\mathbf{M}_{ji}}{\rho_wD_cH_j(t + dt)} \quad (6)$$

Because only four of δV_{ji} and δM_{ji} are non-zero for each i, the sums in (5) and (6) involve 4N terms (not N^2). Actually, there may be less than 4N terms as there is no need to displace and partition cells that are dry.

  This process has time-stepped a wave propagation/flow simulation on a fixed grid while:

(1) conserving water volume exactly (you can verify that the sum of the four non-zero partitioned volumes δV_{ji}, in (3) exactly equal $(H_i D_c^2)$ the volume of the displaced cell.)

(2) conserving linear momentum exactly.

(3) making no special treatment of dry cells or mention of seafloor topography at all for that matter.

(4) reducing a N^2 summation to a 4N one.

(5) obviating the need for a single numerical derivative.

Accelerations: To complete the time step, we must update mean cell accelerations $\mathbf{a}_i(t)$. As is customary in 'long wave' theory, mean acceleration of water in the cell is proportional to the slope of water's upper surface $\zeta(\mathbf{r}_i,t)$

$$\mathbf{a}_i(t) = \mathbf{a}_g(\mathbf{r}_i,t) = -g\nabla_h\zeta(\mathbf{r}_i,t)$$
$$= -g\nabla_h\big[T(\mathbf{r}_i) + H(\mathbf{r}_i,t)\big] \quad (7)$$

$H(\mathbf{r}_i,t)=H_i(t)$ is the water thickness found above, $T(\mathbf{r}_i)$ is the topography taken negative below sea level, and g is the acceleration of gravity, and ∇_h is the horizontal gradient.

The $\nabla_h\zeta(\mathbf{r}_i,t)$ in (7) is the only step where a numerical derivative needs to be evaluated. We avoid even this sole differentiation however, by fitting a plane to several points $\zeta(\mathbf{r}_i,t)$ in the vicinity of $\mathbf{r}_i$ and fixing the horizontal gradient from the slope of that surface. This plane-fitting approach helps stabilize the calculation by estimating the gradient across a two-dimensional region versus adjacent points alone. The size of the region fit to the plane can be adjusted depending upon position. For tsunami propagation for instance, normally you want the horizontal gradient to be averaged over a dimension comparable to the ocean depth. The long wave assumption (that fluid acceleration at all depths in the water column are equal) fails for very short waves. Another advantage of the plane-fitting approach is the ability to "punch out" certain locations near $\mathbf{r}_i$ by excluding them in the fit. Where wet cells are near dry ones say, normally you would "punch out" the dry sites in the calculation of the slope of the fluid surface $\zeta(\mathbf{r}_i,t)$.

Frictions: Additional frictional accelerations can be added to (7) to either slow or drive the system. A dynamic friction that acts in the opposite direction to velocity of the water is common

$$\mathbf{a}_f(\mathbf{r}_i,t) = -c_d\mathbf{v}(\mathbf{r}_i,t)\big|\mathbf{v}(\mathbf{r}_i,t)\big|/H(\mathbf{r}_i,t) \quad (8)$$

The drag coefficient might be a function of space $c_{df} = c_{df}(\mathbf{r}_i)$ to account for dry or wet cells, different vegetation types on land, or to damp waves near the edge of the gird. The coefficient might be a function of time $c_d = c_d(\mathbf{r}_i,t)$ perhaps increasing to stabilize the system at large t.

5) Tsunami Sources.

Given an initial distribution of still water $H(\mathbf{r}_i,t)$, there are several ways to introduce waves.

The first method is Complete Momentum Transfer (CMT). CMT assumes a sudden external transfer of momentum (and possibly mass) to a group of static water cells. CMT is most applicable to high-speed subareal landslides or rockfalls. When these hit water, CMT instantly transfers the momentum of landslide bits to the water in first wet cell that they encounter. From that point onward, that slide bit ceases to exist as far as the simulation is concerned. CMT can produce blast- or impact-like effects.

The second method is No Momentum Transfer (NMT). In NMT, sea bed topography becomes time dependent, i.e. $T(\mathbf{r}_i)$ in (7) goes to $T(\mathbf{r}_i,t)$. NMT vertically lifts up or drops down the water the water in each cell. Gravity acts on the disturbed water surface to make flows or waves, but no momentum is transferred in the lifting itself. NMT has history back to the very beginnings of tsunami simulation. NMT is useful for simulating earthquake tsunami or long run out submarine landslides where wave excitation carries on for many tens of seconds.

The third method we call Drag Along (DA). In DA, external forces are applied to the top or bottom surface of the water in a cell. The force accelerates water much like an anti-friction. For instance, if a submarine landside (or surface wind) was moving at a velocity $\mathbf{v}_s(\mathbf{r}_i,t)$, a drag-along acceleration would be added to (7) as

$$\mathbf{a}_{da}(\mathbf{r}_i,t) = c_{da}\mathbf{v}_s(\mathbf{r}_i,t)\left|\mathbf{v}_s(\mathbf{r}_i,t)\right|/H(\mathbf{r}_i,t) \quad (9)$$

Drag along coefficient c_{da} may or may not equal the dynamic friction coefficient c_{df}. Unlike friction (8) that acts only in the direction opposite to fluid flow, DA (9) can accelerate the flow in any direction.

By employing CMT, NMT and DA either individually or in combination with the theory of Section 4, a single computer program can handle tsunami waves, flows, storm surge, overland flow, dam breaks, inundation and more.